1. (i)*In the group* S_9*, find an element* σ *satisfying* $\sigma(1,3,4,9)(7,2,4)\sigma^{-1} = (8,1,6,3)(5,9,7)$.

   (ii) Prove that if G *is a group in which the map* $x \mapsto x^{-1}$ *is a homomorphism, then* G *must be abelian.*

   OR

   (i) Let G *be any group and* $a, b \in G$*. Prove that* $aba^{-1}b^{-1}$ *can be written as* $aba^{-1}b^{-1} == x^2y^2z^2$ *for some* $x, y, z \in G$*.*

   (ii) Let g *be an element of order* n *in a group* G *and suppose* $xgx^{-1} = g^d$ *for some* $x \in G$*. Prove that* $(n, d) = 1$*.*

   Solution: (i) Order of $(1,3,4,9)(7,2,4)$ is 6 but order of $(8,1,6,3)(5,9,7)$ is 12, therefore, there is no $\sigma \in S_9$ satisfying the mentioned equation.

   (ii) The map $x \mapsto x^{-1}$ is homomorphism i.e. $(xy)^{-1} = x^{-1}y^{-1}$ for all $x, y \in G$.

   Now, taking inverse on both sides we get, $xy = yx$ for all $x, y \in G$ i.e. G is abelian.

   OR

   (i)

   (ii) Here $o(g) = n$ and hence $o(xgx^{-1}) = o(g) = n$.

   Let $(n, d) = m \implies m$ divides n and d both.

   Now, $(xgx^{-1})^{\frac{n}{m}} = g^{d\frac{n}{m}} = e$ because $\frac{d}{m}$ is an integer and $o(g) = n$. But as $\frac{n}{m} \leq n$ and $o(xgx^{-1}) = n$, we conclude that $m = 1$.

2. *If* H *is a subgroup of a group* G *such that each left coset of* H *is equal to some right coset, then prove that* H *is normal in* G*.*

   OR

   Write out an isomorphism between the group G *of eight complex matrices*
   $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}, \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}$
   and the quternion group H *consisting of symbols* $1, i, j, k, -1, -i, -j, -k$ *with the multiplication defined by*

   $$\pm 1.i = \pm i, \pm 1.j = \pm j, \pm 1.k = \pm k,$$
   $$i^2 = j^2 = k^2 = -1, i.j = k, j.i = -k, j.k = i, k.j = -i, k.i = j, i.k = -j$$

   .

   Solution: Let $g_1 \in G$ be arbitrary. Then, we have to show that $g_1Hg_1^{-1} \subseteq H$. Consider, the left coset g_1H. From given condition, there exists $g_2 \in G$ such that $g_1H = Hg_2$ i.e. $g_1Hg_2^{-1} = H$.

   But $g_1 \in g_1H = Hg_2 \implies g_1g_2^{-1} \in H \implies g_1Hg_1^{-1} = g_1Hg_2^{-1}g_2g_1^{-1} \subseteq H$ as $g_1Hg_2^{-1} = H$.

   OR

Define a map that sends $\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$ to 1, $\begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}$ to -1, $\begin{pmatrix}i & 0\\ 0 & -i\end{pmatrix}$ to i, $\begin{pmatrix}0 & i\\ i & 0\end{pmatrix}$ to j, $\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$ to k, $\begin{pmatrix}-i & 0\\ 0 & i\end{pmatrix}$ to $-i$, $\begin{pmatrix}0 & -i\\ -i & 0\end{pmatrix}$ to $-j$ and $\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$ to $-k$.

This map will be a group isomorphism between G and H.

3. *Let G be a non-abelian group of order pq where $p < q$ are primes. Prove that there is a normal subgroup of order q in G.*

   OR

   If $\Gamma_2(n)$ denotes the number of elements in S_n having order ≤ 2, prove that $\Gamma_2(n) = \Gamma_2(n-1) + (n-1)\Gamma_2(n-2)$ for all $n \geq 3$.

   Solution: First of all we show that G has a unique subgroup of order q.

   From Cauchy's theorem there is an element $x \in G$ of order q. Consider $P :=< x >$, cyclic subgroup generated by x. Let there is another subgroup $Q =< y >$ of order q. From Cauchy's theorem, we get $z \in G$ of order p.

   Now, consider $R :=< z >$. As order of PR and QR are pq, therefore $PR = QR$.

   $x \in P \implies x \in PR = QR \implies x = gh$ where $g \in Q, h \in R \implies h = g^{-1}x \in QP \cap R$.

   As, order of QP is p^2, therefore $o(h)$ divides p^2 as well as q i.e. $o(h) = 1$ as p, q are distinct primes and hence $x = g$. Therefore, we get $P \subseteq Q$ and similarly we can show that $Q \subseteq P$ i.e. $P = Q$. Therefore, G has only one subgroup P of order q. Now, let $s \in G$ and consider the subgroup sPs^{-1}. As, the order of sPs^{-1} is q, therefore $sPs^{-1} = P$ because G has a unique subgroup of order q and hence P is normal.

   OR

   In S_n, an element is of order 2 is product of k disjoint transposition where $k \leq n/2$. It is known that S_{n-1} is inside S_n. So, clearly $\Gamma_2(n) \geq \Gamma_2(n-1)$. General element of order ≤ 2 are (1) and $(i_1, j_1)(i_2, j_2) \cdots (i_k, j_k)$ where $i_1, j_1, i_2, j_2, \cdots, i_k, j_k$ are distinct numbers chosen from $S := \{1, 2, \ldots, n-1, n\}$. Number of elements of order ≤ 2 from the set $S' := \{1, 2, \ldots, n-1\}$ is $\Gamma_2(n-1)$ which is including (1). Now, the number of elements of order 2 of the $(1, n)x$ where x is also an element of order 2 from the set $\{1, 2, \ldots, n-2\}$ is $\Gamma_2(n-2)$. Same number will come for (i, n) for all $1 \leq i \leq n-1$. So, $\Gamma_2(n) = \Gamma_2(n-1) + (n-1)\Gamma_2(n-2)$ for $n \geq 3$.

4. *For a subgroup H of a group G, prove that the centralizer*

   $$C_G(H) := \{g \in G : gh = hg \forall h \in H\}$$

   is a normal subgroup of the normalizer

   $$N_G(H) : \{g \in G : gH = Hg\}$$

   .

   OR

   Let G be any group and let $\sigma : G \to G$ be an automorphism. If $Int(g)$ denotes the inner automorphism $x \mapsto gxg^{-1}$ on G for any $g \in G$, show that the composite $\sigma \circ Int(g) \circ \sigma^{-1} = Intt(\sigma(g))$.

   Solution: It is clear that $C_G(H)$ is a subgroup of $N_G(H)$.

   Let, $g \in N_G(H)$ and $g\prime \in C_G(H)$. We have to show that $gg\prime g^{-1} \in C_G(H)$. Let $h \in H$.

Then, $gg'g^{-1}h = gg'h'g^{-1} = gh'g'g^{-1} = hgg'g^{-1}$ where $g^{-1}h = h'g^{-1}$ because $g^{-1}h \in g^{-1}H = Hg^{-1}$ which implies that $C_G(H)$ is a normal subgroup of $N_G(H)$.

OR

Let $x \in G$. Then, $\sigma \circ Int(g) \circ \sigma^{-1}(x) = \sigma(g\sigma^{-1}(x)g^{-1}) = \sigma(g)x\sigma(g)^{-1} = Intt(\sigma(g))(x)$ using the fact that σ is an automorphism. Hence the equality is established.

5. *Give an example of groups $N_1 \leq N_2 \leq N_3$ where N_1 is normal in N_2 and N_2 is normal in N_3 but N_1 is not normal in N_3.*

   OR

   Show that A_4 has no subgroup of order 6.

   Solution:

   Consider $N_3 = D_4 =< a, b : a^4 = b^2 = e, bab = a^{-1} >$, $N_2 =< a^2, b >$ and $N_1 =< b >$. Then, N_1 is normal in N_2, N_2 is normal in N_3 but N_1 is not normal in N_3.

   OR

   $A_4 = \{(1), (1,2)(3,4), (1,3)(2,4), (1,4)(2,3), (1,2,3), (1,3,2), (1,2,4), (1,4,2), (1,3,4), (1,4,3), (2,3,4), (2,4,3)\}$ i.e. A_4 has 3 elements of order 2

   $$(1,2)(3,4), (1,3)(2,4), (1,4)(2,3)$$

   and 8 elements of order 3

   $$(1,2,3), (1,3,2), (1,2,4), (1,4,2), (1,3,4), (1,4,3), (2,3,4), (2,4,3)$$

   . Let H be a subgroup of order 6. Then, using Cauchy's theorem H has an element say (a,b,c) of order 3. So, $(a,b,c)^{-1} = (a,c,b) \in H$. Also, $(1) \in H$. Now, using Cauchy's theorem H has an element of order 2 i.e. one of $(a,b)(c,d), (a,c)(b,d), (a,d)(b,c)$ belongs to H.

   Case 1: Let $(a,b)(c,d) \in H$. Then, $(a,b,c)(a,b)(c,d) = (a,c,d) \in H$ and $(a,c,b)(a,b)(c,d) = (b,c,d) \in H$. Also, $(a,c,d)^{-1} = (a,d,c) \in H$ and $(b,c,d)^{-1} = (b,d,c) \in H$. Therefore, H contains more than 6 elements which is not possible.

   Case 2: Let $(a,c)(b,d) \in H$. Then, $(a,b,c)(a,c)(b,d) = (b,d,c) \in H$ and $(a,c,b)(a,c)(b,d) = (a,b,d) \in H$. Also, $(b,d,c)^{-1} = (b,c,d) \in H$ and $(a,b,d)^{-1} = (a,d,b) \in H$. Therefore, H contains more than 6 elements which is not possible.

   Case 3: Let $(a,d)(b,c) \in H$. Then, $(a,b,c)(a,d)(b,c) = (a,d,b) \in H$ and $(a,c,b)(a,d)(b,c) = (a,d,c) \in H$. Also, $(a,d,b)^{-1} = (a,b,d) \in H$ and $(a,d,c)^{-1} = (a,c,d) \in H$. Therefore, H contains more than 6 elements which is not possible.

   Hence, there is no subgroup of A_4 of order 6.

6. *If G is an abelian group of order p^n, where p is a prime. Show that G has subgroups of order p^r with $r \leq n$.*

   OR

   Let A be the matrix $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Prove that a matrix of the form $\begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix}$ where $y \in \mathbb{R}$, is expressible as $DAD^{-1}A^{-1}$ for some diagonal matrix D of the form $\text{diag}(t, t^{-1})$ with $t \in \mathbb{R}^*$ if and only if $y \in (-1, \infty)$.

   Solution: See the proof of Sylow's first theorem from any standard book.

OR

Here, $A = \begin{pmatrix} t & 1 \\ 0 & t^{-1} \end{pmatrix}$ and let $D = \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix}$ where $t \in \mathbb{R}^*$. Then $DAD^{-1}A^{-1} = \begin{pmatrix} 1 & t^2 - 1 \\ 0 & 1 \end{pmatrix}$. Therefore, $\begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix}$ where $y \in \mathbb{R}$, is expressible as $DAD^{-1}A^{-1}$ if and only if $y \in (-1, \infty)$ as $t^2 - 1 \in (-1, \infty)$ for $t \in \mathbb{R}^*$.